Last update: 11:50pm, Jan 13.

Math 451: Functional Analysis | Homework 1 | **Due: Jan ??, 23:59**

1. Let $(X, \mathcal{B})$ be a measurable space, and let $M(X, \mathcal{B})$ be the space of finite complex measures on $(X, \mathcal{B})$. Then $\|\mu\| = |\mu|(X)$ is a norm on $M(X)$ that makes $M(X)$ into a Banach space.

   Hint: Use the absolutely convergent series criterion for completeness.

2. Let $C^k([0,1])$ be the space of functions on $[0,1]$ possessing continuous derivatives up to order k on $[0,1]$, including one-sided derivatives at the endpoints. Prove that

$$\|f\| = \sum_{j=0}^{k} \left\|f^{(j)}\right\|_u$$

   is a norm on $C^k([0,1])$ that makes $C^k([0,1])$ into a Banach space.

   Hint: The main point is to show that if $\{f_n\} \subset C^1([0,1])$ with $f_n \to f$ and $f_n' \to g$ uniformly, then $f' = g$. Use the fundamental theorem of calculus.

3. Let $L_k^1([0,1])$ be the space of all $f \in C^{k-1}([0,1])$ such that $f^{(k-1)}$ is absolutely continuous on $[0,1]$ (and hence $f^{(k)}$ exists a.e. and is in $L^1([0,1])$). Then

$$\|f\| = \sum_{j=0}^{k} \int_0^1 \left|f^{(j)}(x)\right| dx$$

   is a norm on $L_k^1([0,1])$ that makes $L_k^1([0,1])$ into a Banach space.

   Hint: Fundamental theorem of calculus. Consult Lecture 25 of Math 564, Fall 2025.

4. This exercise demonstrates two opposite situations of the relationships of the L^p spaces: large scale (at infinity) and small scale. Let $0 < p \leqslant q \leqslant \infty$. Prove:

   (a) For any set X, we have $\ell^p(X) \subseteq \ell^q(X)$; in fact, $\|f\|_q \leqslant \|f\|_p$ for all $f : X \to \mathbb{C}$.

   (b) For any finite measure space (X, μ), we have $L^q(\mu) \subseteq L^p(\mu)$; in fact, $\|f\|_p \leqslant \|f\|_q \cdot \mu(X)^{\frac{1}{p} - \frac{1}{q}}$. In particular, when (X, μ) is a probability space, we have $\|f\|_p \leqslant \|f\|_q$.

MORE QUESTIONS TO BE ADDED.